Quasi-isometric rigidity for $PSL_2(\mathbf{Z}[\frac{1}{p}]$

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Abstract

We prove that $PSL_2(\mathbf{Z}[\frac{1}{p}])$ gives the first example of groups which are not quasi-isometric to each other but have the same quasi-isometry group. Namely, $PSL_2(\mathbf{Z}[\frac{1}{p}])$ and $PSL_2(\mathbf{Z}[\frac{1}{q}])$ are not quasi-isometric unless $p = q$, and, independent of p, the quasi-isometry group of $PSL_2(\mathbf{Z}[\frac{1}{p}])$ is $PSL_2(\mathbf{Q})$. In addition, we characterize $PSL_2(\mathbf{Z}[\frac{1}{p}])$ uniquely among all finitely generated groups by its quasi-isometry type.

1 Introduction

Combining the work of many people yields a complete quasi-isometry classification of irreducible lattices in semisimple Lie groups (see [F] for an overview of these results). One of the first general results in this classification is the complete description, up to quasi-isometry, of all nonuniform lattices Λ in semisimple Lie groups of rank 1, proved by R. Schwartz [S2]. He shows that every quasi-isometry of such a lattice Λ is equivalent to a unique commensurator of Λ. (A *commensurator* of $\Lambda \subset G$ is an element $g \in G$ so that $g\Lambda g^{-1} \cap \Lambda$ has finite index in Λ.) We will call this result *commensurator rigidity*, although it is a different notion than the commensurator rigidity of Margulis. In [FS] it was conjectured that commensurator rigidity, or at least a slightly weaker statement, "quasi-isometric iff commensurable," should apply to nonuniform lattices in a wide class of Lie groups. Here we prove that both of these statements are true for $PSL_2(\mathbf{Z}[\frac{1}{p}])$.

In a different direction, B. Farb and L. Mosher proved analogous quasi-isometric rigidity results for the the solvable Baumslag-Solitar groups. These groups are given by the presentation

$$BS(1,n) = < a, b | aba^{-1} = b^n >$$

and are not lattices in any Lie group.

The group $PSL_2(\mathbf{Z}[\frac{1}{p}])$ is a nonuniform (i.e. non cocompact) lattice in $PSL_2(\mathbf{R}) \times PSL_2(\mathbf{Q}_p)$, analogous to the classical Hilbert modular group $PSL_2(\mathcal{O}_d)$ in $PSL_2(\mathbf{R}) \times PSL_2(\mathbf{R})$. It is also a basic example of an S-arithmetic group. The proofs of Theorems A, B and C (stated below) combine techniques from the two types of quasi-isometric rigidity results mentioned above. When we construct a space Ω_p on which $PSL_2(\mathbf{Z}[\frac{1}{p}])$ acts properly discontinuously and cocompactly by isometries, we see that the horospheres forming the boundary components of Ω_p carry the geometry of the group $BS(1,p)$. In this way the results of [FM] play a role in the quasi-isometric rigidity of $PSL_2(\mathbf{Z}[\frac{1}{p}])$.

1.1 Statement of Results

In this paper we prove the following quasi-isometric rigidity results for the finitely generated groups $PSL_2(\mathbf{Z}[\frac{1}{p}])$, where p is a prime. Theorem A may be viewed as a strengthening of strong (Mostow) rigidity for $PSL_2(\mathbf{Z}[\frac{1}{p}])$. [M]

Theorem A (Main Theorem). *Every quasi-isometry of $PSL_2(\mathbf{Z}[\frac{1}{p}])$ is equivalent to a commensurator of $PSL_2(\mathbf{Z}[\frac{1}{p}])$. Hence the natural map*

$$Comm(PSL_2(\mathbf{Z}[\tfrac{1}{p}])) \to QI(PSL_2(\mathbf{Z}[\tfrac{1}{p}]))$$

is an isomorphism.

Since for any prime p the commensurator group of $PSL_2(\mathbf{Z}[\frac{1}{p}])$ in $PSL_2(\mathbf{R}) \times PSL_2(\mathbf{Q}_p)$ is $PSL_2(\mathbf{Q})$, the quasi-isometry group is also $PSL_2(\mathbf{Q})$. (See §2.2 for a model of PSL_2 as an algebraic group.) Thus we cannot distinguish the quasi-isometry classes of these groups via their quasi-isometry groups. However, using a result of B. Farb and L. Mosher [FM] (Theorem 2.1 below), we are able to prove the following.

Theorem B (Quasi-isometric iff commensurable). *Let p and q be primes. Then $PSL_2(\mathbf{Z}[\frac{1}{p}])$ and $PSL_2(\mathbf{Z}[\frac{1}{q}])$ are quasi-isometric if and only if they are commensurable, which occurs only when $p = q$.*

Theorems A and B together give the first example of groups which have the same quasi-isometry group but are not quasi-isometric.

The following theorem characterizes $PSL_2(\mathbf{Z}[\frac{1}{p}])$ uniquely among all finitely generated groups by its quasi-isometry type.

Theorem C (Quasi-isometry characterization). *Let Γ be any finitely generated group. If Γ is quasi-isometric to $PSL_2(\mathbf{Z}[\frac{1}{p}])$, then there is a short exact sequence*

$$1 \to N \to \Gamma \to \Lambda \to 1$$

where N is a finite group and Λ is abstractly commensurable to $PSL_2(\mathbf{Z}[\frac{1}{p}])$.

Two groups are *abstractly commensurable* if they have isomorphic finite index subgroups.

1.2 An outline of the proofs of Theorems A and B

The group $PSL_2(\mathbf{Z}[\frac{1}{p}])$ is a nonuniform (i.e. non-cocompact) lattice in $G = PSL_2(\mathbf{R}) \times PSL_2(\mathbf{Q}_p)$ under the diagonal embedding sending a matrix M to the pair (M, M). The group G acts on $\mathbf{H}^2 \times T_p$, where $\mathbf{H}^2$ is the hyperbolic plane and T_p is the Bruhat-Tits-Serre tree associated to $PGL_2(\mathbf{Q}_p)$. Let $f : PSL_2(\mathbf{Z}[\frac{1}{p}]) \to PSL_2(\mathbf{Z}[\frac{1}{q}])$ be a quasi-isometry. The proofs of Theorems A and B both begin as follows.

Step 1 (The geometric model). We construct a space Ω_p on which $PSL_2(\mathbf{Z}[\frac{1}{p}])$ acts properly discontinuously and cocompactly by isometries (hence by a result of Milnor and Svarc [Mi], $PSL_2(\mathbf{Z}[\frac{1}{p}])$ and Ω_p are quasi-isometric). The space Ω_p has a boundary consisting of horospheres of $\mathbf{H}^2 \times T_p$, each of which is a quasi-isometrically embedded copy of the group $BS(1,p)$. The quasi-isometry $f : PSL_2(\mathbf{Z}[\frac{1}{p}]) \to PSL_2(\mathbf{Z}[\frac{1}{q}])$ then induces a quasi-isometry, also denoted f, from Ω_p to Ω_q.

Step 2 (The Boundary Detection Theorem). This theorem shows that for every horosphere boundary component of $\partial\Omega_p$, there is a corresponding horosphere boundary component of $\partial\Omega_q$ so that f restricts to a quasi-isometry of horospheres. The proof of this theorem uses the Coarse Separation Theorem of [FS] and [S2], and the geometry of Ω_p.

Remark. We are now able to prove Theorem B. The initial quasi-isometry $f : PSL_2(\mathbf{Z}[\frac{1}{p}]) \to PSL_2(\mathbf{Z}[\frac{1}{q}])$ induces a quasi-isometry $f : \Omega_p \to \Omega_q$. From the Boundary Detection Theorem, we obtain a quasi-isometry $\hat{f} : \sigma \to \tau$ by restriction, where σ and τ are horosphere boundary components of Ω_p and Ω_q, respectively. By Step 1, the map $\hat{f}$ can be considered as a quasi-isometry of Baumslag-Solitar groups, namely $\hat{f} : BS(1,p) \to BS(1,q)$. From Theorem 2.1 [FM] we conclude that $p = q$.

The proof of Theorem A continues with the following steps. We are now considering a quasi-isometry $f : \Omega_p \to \Omega_p$.

Step 3 (The geometry of Ω_p). For any two horosphere boundary components σ_1 and σ_2 of Ω_p, there is a unique line $l \subset T_p$ so that σ_1 and σ_2 are a specified fixed distance apart in $\mathbf{H}^2 \times t$, for all vertices

t of l. This line is called the *closeness line* of σ_1 and σ_2. The set of closeness lines of all horospheres is preserved under quasi-isometry. This geometric result replaces the usual group-equivariance assumed in Mostow-Prasad rigidity, and provides the structure necessary for Step 4.

Step 4 (S-Arithmetic Action Rigidity). Here we prove an 2 dimensional S-arithmetic version of the Action Rigidity Theorem of R. Schwartz [S1]. The Action Rigidity Theorem concludes that the map induced by f on the set of horospheres of $\partial\Omega_p$ (which is indexed by $\mathbf{Q} \cup \{\infty\}$) is given by an affine map of $\mathbf{Q} \cup \{\infty\}$.

Step 5 (Conclusion of the proof). From Step 4, we are able to choose a specific commensurator g of $PSL_2(\mathbf{Z}[\frac{1}{p}])$ so that the composite map $f \circ g$ is a bounded distance from the identity map. This finishes the proof of Theorem A.

2 Preliminary material

2.1 Quasi-isometries

Definition. Let $K \geq 1$ and $C \geq 0$. A (K, C)-*quasi-isometry* between metric spaces (X, d_X) and (Y, d_Y) is a map $f : X \to Y$ satisfying:

1. $\frac{1}{K} d_X(x_1, x_2) - C \leq d_Y(f(x_1), f(x_2)) \leq K d_X(x_1, x_2) + C$ for all $x_1, x_2 \in X$.

2. For some constant C', the C' neighborhood of $f(X)$ is all of Y.

We often omit the constants K and C and simply refer to f as a quasi-isometry. A quasi-isometry f can always be changed by a bounded amount using the standard "connect-the-dots" procedure so that it is continuous. (See, e.g. [FS].) A quasi-isometry also has a coarse inverse, i.e. there is a quasi-isometry $g : Y \to X$ so that $f \circ g$ and $g \circ f$ are a bounded distance from the appropriate identity map in the sup norm. A map satisfying 1. but not 2. in the definition above is called a *quasi-isometric embedding*.

We define the *quasi-isometry group* of a space X, denoted $QI(X)$, to be the set of all self quasi-isometries of X, modulo those a bounded distance from the identity in the sup norm, under composition of quasi-isometries. Inverses exist in $QI(X)$ since every quasi-isometry has a coarse inverse. A quasi-isometry between two metric spaces X and Y induces an isomorphism between $QI(X)$ and $QI(Y)$.

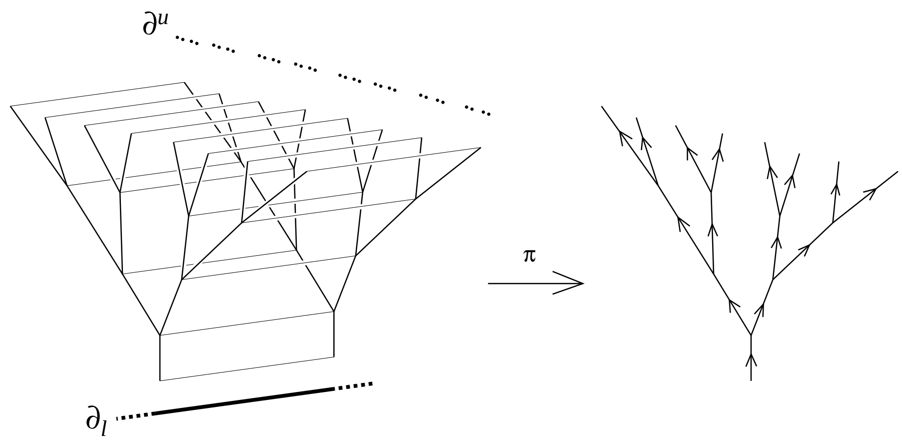

Figure 1: The complex X_n associated to $BS(1,n)$ where the map π denotes projection onto the tree factor. The upper and lower boundaries are also marked.

2.2 PSL_2 as a algebraic group

We will use the following model of PSL_2 as an algebraic group. Consider the map $Ad : SL_2(\mathbf{C}) \to GL(sl_2)$, where we view the Lie algebra sl_2 as a vector space. Let $G' = Ad(SL_2(\mathbf{C}))$. Then G' is a model for PSL_2 as an algebraic group, since the center of $SL_2(\mathbf{C})$ vanishes under the map Ad. By $PSL_2(\mathbf{Q})$ we mean the $\mathbf{Q}-$points of G', denoted $G'_{\mathbf{Q}}$.

2.3 The geometry of $BS(1,n)$

The Baumslag-Solitar group $BS(1,n)$ acts properly discontinuously and cocompactly by isometries on a metric $2-$complex X_n defined explicitly in [FM]. This complex X_n is topologically $T_n \times R$, where T_n is a regular $(n+1)$-valent tree, directed so that each vertex has 1 incoming edge and n outgoing edges. (Figure 1.)

A *height function* on T_n is a continuous function $h : T_n \to \mathbf{R}$ which maps each oriented edge of T_n homeomorphically onto an oriented interval of a given length d. A vertex of T_n whose height under h is kd is said to have *combinatorial height* k. Fix a basepoint for T_n with height 0. This determines a height function on T_n. Let $\pi : T_n \times \mathbf{R} \to T_n$ denote projection. Then $h \circ \pi$ is a height function on X_n.

A *proper line* in T_n is the image of a proper embedding $\mathbf{R} \to T_n$. A coherently oriented proper line is one on which the height function is strictly monotone. We will use the term "line" to mean a proper line in T_n. The metric on X_n is defined so that for each infinite, coherently

oriented line $l \subset T$, the plane $l \times \mathbf{R}$ is isometric to a hyperbolic plane.

When studying the geometry of X_n, there are two "boundaries" of the complex which play an important role. (Figure 1.) The *lower boundary*, denoted $\partial_l X_n$, is homeomorphic to $\mathbf{R}$ and is the common lower boundary of all hyperbolic planes in X_n.

The *upper boundary*, denoted $\partial^u X_n$, is defined to be the space of hyperbolic planes in X_n, with the following metric. If Q_1 and Q_2 are hyperbolic planes in X_n which agree below combinatorial height k, define the distance between them to be n^{-k}. With this metric, $\partial^u X_n$ is isometric to the set of $n-$adic rational numbers, $\mathbf{Q}_n$, with the metric defined by the $n-$adic absolute value.

B. Farb and L. Mosher obtain the following quasi-isometric rigidity results for $BS(1,n)$. [FM]

Theorem 2.1 ([FM]). *For integers $m, n \geq 2$, the groups $BS(1,m)$ and $BS(1,n)$ are quasi-isometric if and only if they are commensurable. This happens if and only if there exist integers $r, j, k > 0$ such that $m = r^j$ and $n = r^k$.*

Theorem 2.2 ([FM]). *The quasi-isometry group of $BS(1,n)$ is given by the following isomorphism:*

$$QI(BS(1,n)) \cong Bilip(\mathbf{R}) \times Bilip(\mathbf{Q}_n).$$

A quasi-isometry $f \in QI(BS(1,n))$ induces bilipschitz maps f^u and f_l on the upper and lower boundaries of X_n, respectively [FM]. From Theorem 2.2 we see that the map $QI(BS(1,n)) \to Bilip(\mathbf{R}) \times Bilip(\mathbf{Q}_p)$ given by $f \to (f_l, f^u)$ is an isomorphism. It is perhaps surprising that $PSL_2(\mathbf{Z}[\frac{1}{p}])$ should have such a small quasi-isometry group while $BS(1,p)$ has such a large quasi-isometry group.

3 The geometry of $PSL_2(\mathbf{Z}[\frac{1}{p}])$

3.1 The action of $PSL_2(\mathbf{R}) \times PSL_2(\mathbf{Q}_p)$ on $\mathbf{H}^2 \times T_p$

We will consider $PSL_2(\mathbf{Z}[\frac{1}{p}]) \subset PSL_2(\mathbf{R}) \times PSL_2(\mathbf{Q}_p)$ as the image of the diagonal map $\eta : PSL_2(\mathbf{Z}[\frac{1}{p}]) \to PSL_2(\mathbf{R}) \times PSL_2(\mathbf{Q}_p)$ given by $\eta(M) = (M, M)$. Viewed in this way, $PSL_2(\mathbf{Z}[\frac{1}{p}])$ is a lattice in the group $PSL_2(\mathbf{R}) \times PSL_2(\mathbf{Q}_p)$, for any prime p.

We now define the Bruhat-Tits tree T_p associated to $PGL_2(\mathbf{Q}_p)$. We consider a tree T to be a set of vertices $Vert(T)$ together with a set of adjacency relations among the vertices. Let $Vert(T_p)$ be the set of equivalence classes of $\mathbf{Z}_p$-lattices in $\mathbf{Q}_p \times \mathbf{Q}_p$. Two lattices L_1 and L_2 are equivalent if $L_2 = \alpha L_1$, where $\alpha \in \mathbf{Q}_p - \{0\}$. Two vertices $[L_1]$ and

$[L_2]$ are adjacent if there exist representatives $L_1 \in [L_1]$ and $L_2 \in [L_2]$ with $L_1 \subset L_2$ and $[L_2 : L_1] = p$. An example of two adjacent vertices is $[\mathbf{Z}_p \times \mathbf{Z}_p]$ and $[p\mathbf{Z}_p \times \mathbf{Z}_p]$. The tree T_p is a regular $(p+1)$-valent tree. We will fix $[L_0] = [\mathbf{Z}_p \times \mathbf{Z}_p]$ as the basepoint of T_p as well as a height function h giving $[L_0]$ height 0. An element $g \in PGL_2(\mathbf{Q}_p)$ acts on $[L] \in Vert(T_p)$ by matrix multiplication on the basis vectors of a representative lattice in the equivalence class $[L]$. Note that $PSL_2(\mathbf{Q}_p)$ also acts on T_p in this way.

Let $\mathbf{H}^2$ denote $2-$dimensional hyperbolic space in the upper half plane model, i.e. $\mathbf{H}^2 = \{(x,y)|x \in \mathbf{R}, y > 0\}$ with the metric $\frac{dx^2+dy^2}{y^2}$. We define the action of an element $(g_1, g_2) \in PSL_2(\mathbf{R}) \times PSL_2(\mathbf{Q}_p)$ on a point $(x, [L]) \in \mathbf{H}^2 \times T_p$. The element $g_1 \in PSL_2(\mathbf{R})$ acts on $x \in \mathbf{H}^2$ by fractional linear transformations. The element $g_2 \in PSL_2(\mathbf{Q}_p)$ acts on $[L] \in Vert(T_p)$ by matrix multiplication on the basis vectors of a representative lattice in the equivalence class $[L]$. When we are considering the action of $PSL_2(\mathbf{Z}[\frac{1}{p}])$ on $\mathbf{H}^2 \times T_p$, we have $g_1 = g_2$. We will use only one coordinate to represent the elements of $PSL_2(\mathbf{Z}[\frac{1}{p}]) \subset PSL_2(\mathbf{R}) \times PSL_2(\mathbf{Q}_p)$. So $g \in PSL_2(\mathbf{Z}[\frac{1}{p}])$ would correspond to $(g,g) \in PSL_2(\mathbf{R}) \times PSL_2(\mathbf{Q}_p)$. Hence we can refer to $g \in PSL_2(\mathbf{Z}[\frac{1}{p}])$ acting on either $\mathbf{H}^2$ or T_p or $\mathbf{H}^2 \times T_p$ in the appropriate manner.

3.2 Constructing the space Ω_p

We want to construct a space $\Omega_p \subset \mathbf{H}^2 \times T_p$ on which $PSL_2(\mathbf{Z}[\frac{1}{p}])$ acts properly discontinuously and cocompactly by isometries. The Milnor-Svarc criterion states that if a finitely generated group Γ acts properly discontinuously and cocompactly by isometries on a space X, then Γ is quasi-isometric to X. We then refer to the geometry of X as the large scale geometry of Γ. This additional geometric information associated to Γ is often useful in determining rigidity properties of Γ.

Although $PSL_2(\mathbf{Z}[\frac{1}{p}])$ acts by isometries on $\mathbf{H}^2 \times T_p$, it does not act cocompactly, because the fundamental domain for the action of $PSL_2(\mathbf{Z}[\frac{1}{p}])$ on a fixed $\mathbf{H}^2$ is the same as the fundamental domain for the action of $PSL_2(\mathbf{Z})$ on $\mathbf{H}^2$, which is unbounded in one direction.

Let w be the segment of the horocircle based at ∞ of height h_0 in this fundamental domain (in the upper half space model of hyperbolic space), for h_0 sufficiently large. Fix $H > 1$. Lift the segment w to $\mathbf{H}^2 \times [L_0]$ to obtain a horocyclic segment at height H whose orbit under $PSL_2(\mathbf{Z}) = Stab_{PSL_2(\mathbf{Z}[\frac{1}{p}])}([L_0])$ is a disjoint collection of horocircles of $\mathbf{H}^2$, centered at $\mathbf{Q} \cup \{\infty\} \subset \partial_\infty \mathbf{H}^2$. The orbit of the lift of w under the entire group $PSL_2(\mathbf{Z}[\frac{1}{p}])$ gives a $PSL_2(\mathbf{Z}[\frac{1}{p}])$-equivariant collection of horocircles in $\mathbf{H}^2 \times T_p$, based at $\mathbf{Q} \cup \{\infty\}$ in each copy of

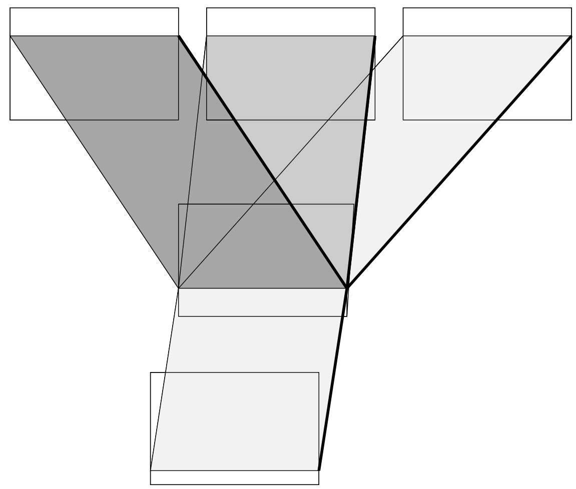

Figure 2: This is a piece of the horosphere σ_∞ in $\mathbf{H}^2 \times T_3$, the space associated to $PSL_2(\mathbf{Z}[\frac{1}{3}])$. Notice how the height of the horosphere increases in each successive copy of $\mathbf{H}^2$. Viewing the bold black lines as part of the tree T_3 helps one to see that this horosphere is topologically $T_3 \times \mathbf{R}$.

$\mathbf{H}^2 \subset \mathbf{H}^2 \times T_p$.

We now define a *horosphere* of $\mathbf{H}^2 \times T_p$ based at $\alpha \in \mathbf{R} \cup \{\infty\}$ to be the collection of horocircles, one in each $\mathbf{H}^2 \times [L]$ for every $[L] \in Vert(T_p)$, all based at α. According to the above construction, there is exactly one such horocircle in each $\mathbf{H}^2 \times [L]$. We will denote this horosphere of $\mathbf{H}^2 \times T_p$ by σ_α.

In order to have a "connected" picture of a horosphere, we can put an edge e between any two adjacent vertices of T_p and extend the horosphere linearly in $\mathbf{H}^2 \times e$. So we can think of a horosphere as a hollow tube, or in the case of σ_∞ as a flat sheet, whose image under the projection $\pi : \mathbf{H}^2 \times T_p \to T_p$ is all of T_p. (Figures 2 and 3.)

We define the space Ω_p, where $PSL_2(\mathbf{Z}[\frac{1}{p}])$ acts properly discontinuously and cocompactly by isometries, to be $\mathbf{H}^2 \times T_p$ with the interiors of all the horospheres removed. The interior of a horosphere is the union of the interiors of the component horocircles.

3.3 The metric

The following theorem allows us to use the product metric $d_{\mathbf{H}^2} \times d_T$ on Ω_p. Although it is stated for semisimple Lie groups, it is proven in the more general context of S-arithmetic groups.

Theorem 3.1. *[LMR] If G is a semisimple Lie group of rank at least 2 and Γ is an irreducible lattice in G then, d_R restricted to Γ is Lipschitz equivalent to d_W, where d_W is the word metric on Γ and d_R is the left invariant Riemannian metric on Γ.*

By construction, $PSL_2(\mathbf{Z}[\frac{1}{p}])$ acts properly discontinuously and cocompactly by isometries on Ω_p.

3.4 The closeness line

We are now interested in the packing of the horospheres of $\mathbf{H}^2 \times T_p$, i.e. how the horosphere "tubes" fit together. We consider two horospheres, without loss of generality σ_0 and σ_∞, and the distance between them when restricted to $\mathbf{H}^2 \times [L]$, for any $[L] \in Vert(T_p)$. We use the notation $\sigma|_{[L]}$ for the horosphere $\sigma \subset \partial\Omega_p$ restricted to $\mathbf{H}^2 \times [L]$. We will show that there is a unique line $l \subset T_p$ so that σ_0 and σ_∞ remain a constant distance apart when restricted to $\mathbf{H}^2 \times [L]$, for all vertices $[L]$ lying on l. In addition, we will show that all other lines $l' \subset T_p$ have the property that $d_{\mathbf{H}^2}(\sigma_0|_{[L]}, \sigma_\infty|_{[L]})$ increases without bound as the vertices $[L]$ lying on l' increase in height. We call l the *closeness line* of σ_∞ and σ_0. By symmetry, any two horospheres σ_α and σ_β have a unique closeness line.

We will use the notation $\sigma|_{[L]}$ for the horosphere σ of $\partial\Omega_p$ restricted to $\mathbf{H}^2 \times [L]$, for $[L] \in Vert(T_p)$. Note that a matrix g which translates $\sigma_\alpha|_{[L_1]}$ to $\sigma_\alpha|_{[L_2]}$ will lie in $Stab_{PSL_2(\mathbf{Z}[\frac{1}{p}])}(\alpha)$ and satisfy $[gL_1] = [L_2]$. However, to move between the horocircles $\sigma_\beta|_{[L_1]}$ and $\sigma_\beta|_{[L_2]}$, for $\beta \neq \alpha$, we will need to use a different element of $PSL_2(\mathbf{Z}[\frac{1}{p}])$, i.e. $h \in Stab_{PSL_2(\mathbf{Z}[\frac{1}{p}])}(1)$ such that $[hL_1] = [L_2]$.

Products of the matrices $A = \begin{pmatrix} p & 0 \\ 0 & \frac{1}{p} \end{pmatrix}$ and $B = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$ give the $PSL_2(\mathbf{Q}_p)$ action on T_p. (See, e.g. [Se].) Since

$$A \in Stab_{PSL_2(\mathbf{Z}[\frac{1}{p}])}(0) \cap Stab_{PSL_2(\mathbf{Z}[\frac{1}{p}])}(\infty),$$

letting A^i act on $\mathbf{H}^2 \times [L_0]$ moves the horocircle $\sigma_0|_{[L_0]}$ (resp. $\sigma_\infty|_{[L_0]}$) to the horocircle $\sigma_0|_{[A^iL_0]}$ (resp. $\sigma_\infty|_{[A^iL_0]}$). Moreover, $A \in PSL_2(\mathbf{R}) = Isom^+(\mathbf{H}^2)$, so we have

$$d_{\mathbf{H}^2}(\sigma_0|_{[L_0]}, \sigma_\infty|_{[L_0]}) = d_{\mathbf{H}^2}(\sigma_0|_{[A^iL_0]}, \sigma_\infty|_{[A^iL_0]}) = 2\log H$$

where H was chosen in §3.2. Let l be the line in T_p which is the orbit of $[L_0]$ under the cyclic group generated by the matrix A. We call l the *diagonal line*.

Recall that we usually need two different matrices to move two different horocircles in $\mathbf{H}^2 \times [L_0]$ to their corresponding horocircles in

$\mathbf{H}^2 \times [L]$ for $[L] \in Vert(T_p)$; here we need only one matrix since the matrix A lies in the intersection of the stabilizers of 0 and ∞.

We will view a ray of T_p based at $[L_0]$ as an infinite sequence of products $\{\Pi_{i=1}^N C_i\}_{N \in \mathbf{N}}$ where either $C_i = A$ or $C_i = B^j A$. The diagonal ray (i.e. the part of the diagonal line beginning at $[L_0]$ and moving upwards in height) is described by $\{\Pi_{i=1}^N A\}_{N \in \mathbf{N}}$, and we know from the previous paragraph that $d_{\mathbf{H}^2}(\sigma_0|_{[\Pi_{i=1}^N A]}, \sigma_\infty|_{[\Pi_{i=1}^N A]}) = 2 \log H$ for all $N \in \mathbf{N}$.

Now suppose that $r \subset T_p$ is the ray based at $[L_0]$ given by the sequence $\{\Pi_{i=1}^N C_i\}_{N \in \mathbf{N}}$ where $C_1 = B^j A$ for $j \in \{0, 1, \cdots p-1\}$. For any $N_0 \in N$, the product $\Pi_{i=1}^N C_i$ has the form $(\Pi_{i=1}^T A)(\Pi_{i=1}^S C_i)$ where C_i is as above. Such a product is a matrix of the form $M = \begin{pmatrix} p^{S+T} & \frac{s}{p^{S-T}} \\ 0 & \frac{1}{p^{S+T}} \end{pmatrix}$, where S or T may be 0. We may assume that $S > T$, meaning that in the tree factor, we are considering vertices sufficiently far away from the diagonal line l.

Let $\beta = \frac{-s}{p^{2S}}$, where s and S are as in the matrix M. We can find a matrix of the form $N = \begin{pmatrix} a & -s \\ b & p^{2S} \end{pmatrix} \in PSL_2(\mathbf{Z})$ where s, S are as in M. Since $N \in PSL_2(\mathbf{Z}) = Stab_{PSL_2(\mathbf{Z}[\frac{1}{p}])}([L_0])$ and $N \cdot 0 = \beta$, the matrix N maps $\sigma_0|_{[L_0]}$ to $\sigma_\beta|_{[L_0]}$. A computation shows that the height of $\sigma_\beta|_{[L_0]}$ is $\frac{1}{p^{2(S+T)}H}$. Moreover, $M \cdot \beta = 0$, so $M \cdot (\sigma_\beta|_{[L_0]}) = \sigma_0|_{[ML_0]}$. Another computation shows that $\sigma_0|_{[ML_0]}$ has height $\frac{1}{p^{2m}H}$ in $\mathbf{H}^2$. The height of $\sigma_\infty|_{[ML_0]}$ is $p^{2(S+T)}H$. Hence

$$d_{\mathbf{H}^2}(\sigma_0|_{[ML_0]}, \sigma_\infty|_{[ML_0]}) = \log(p^{2S}H^2).$$

When $S = 0$, the vertex $\Pi_1^T A$ lies on the diagonal line, and the above formula gives a constant distance of $2 \log H$ between σ_∞ and σ_0 in $A \times l$. When $S \neq 0$, we see that S increases by a factor of 2 for each unit of height in T_p; hence the distance between σ_∞ and σ_0 increases by a factor of p^4.

So we see that for any line $l' \subset T$ which is not the diagonal line, $d_{\mathbf{H}^2}(\sigma_0|_{[L]}, \sigma_\infty|_{[L]})$ increases without bound as we consider vertices $[L]$ of l' of increasing height. So the diagonal line l is the closeness line of σ_0 and σ_∞.

It is clear by construction that the closeness line of σ_α and σ_β and the closeness line of σ_α and σ_γ are distinct.

3.5 The geometry of the horospheres

Consider the matrices $A = \begin{pmatrix} p & 0 \\ 0 & \frac{1}{p} \end{pmatrix}$ and $B = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$. Then $A, B \in PSL_2(\mathbf{Z}[\frac{1}{p}]) \subset PSL_2(\mathbf{Q}_p)$ and

$$ABA^{-1} = \begin{pmatrix} 1 & p^2 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}^{p^2} = B^{p^2}.$$

The map Φ sending the generator a of $BS(1,p^2)$ to the matrix A and the generator b to B is a homomorphism of $BS(1,p^2)$ into the group $PSL_2(\mathbf{Q}_p)$. Products of the matrices A and B form the elements of $PGL_2(\mathbf{Q}_p)$ needed to move between vertices of T_p.

If we take the orbit of the segment w from Hi to $1+Hi$ in $\mathbf{H}^2 \times [L_0]$ under the group $\Phi(BS(1,p^2)) \subset PSL_2(\mathbf{Q}_p)$, we obtain the horosphere σ_∞. The width of a horostrip in σ_∞ between vertices whose combinatorial heights differ by 1, in the metric on $\mathbf{H}^2 \times T_p$, is $1 + 2\log p$, while in X_{p^2} it is $2 \log p$. Since these distances are comparable, the horosphere σ_∞ (and hence any horosphere σ_α) is quasi-isometric to the complex X_{p^2} associated to $BS(1,p^2)$. Since $BS(1,p^2)$ and $BS(1,p)$ are commensurable, hence quasi-isometric, we may assume that the horospheres are quasi-isometrically embedded copies of the complex X_p associated to $BS(1,p)$.

3.6 Another view of the closeness lines

The following discussion of closeness lines provides some geometric intuition for understanding these lines but is not necessary for the proofs of Theorems A, B and C.

We can consider the collection of closeness lines of all horospheres of $\partial\Omega_p$ with the horosphere σ_∞. This is a collection of distinct lines in the tree T_p. These closeness lines can also be viewed as lying in the horosphere σ_∞, which is thought of as the complex X_p associated to $BS(1,p)$. In this case, the closeness line of σ_∞ and σ_α can be viewed as the set of points in σ_∞ closest to σ_α which project under $\pi : \mathbf{H}^2 \times T_p \to T_p$ to the closeness line of σ_∞ and σ_α as described in §3.4. (Figure 4.) Then each hyperbolic plane $\mathbf{H}^2 \subset X_p = \sigma_\infty$ intersects these closeness lines, with at most one line lying completely in that plane. (Figure 5.)

4 The Boundary Detection Theorem

The first goal of this section is to prove Theorem 4.5 (Boundary Detection Theorem), which states that a quasi-isometry $f : \Omega_p \to \Omega_q$ maps a horosphere boundary component of Ω_p to within a bounded Hausdorff

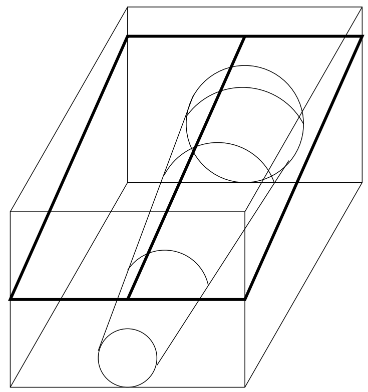

Figure 3: The two horospheres shown are σ_α and σ_∞, for $\alpha \neq \infty$. The dark line in σ_∞ represents the closest points in σ_∞ to σ_α. We can view this line as the closeness line of σ_∞ and σ_α drawn in σ_∞.

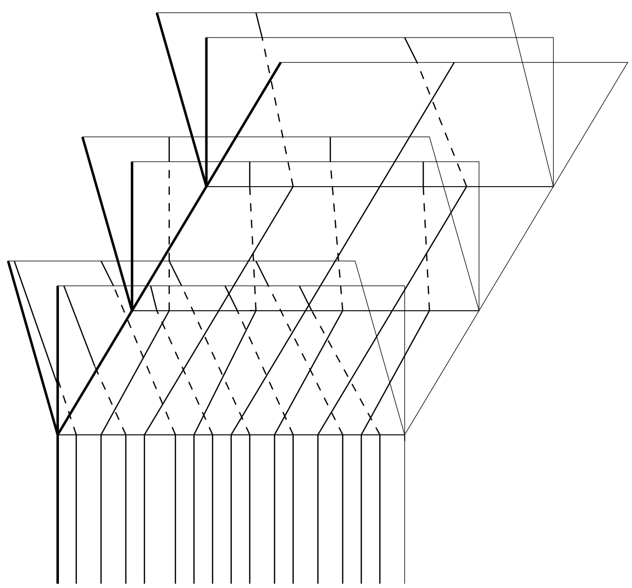

Figure 4: A subset of the collection of closeness lines of all horospheres with σ_∞, drawn in the complex X_p of σ_∞.

distance of a horosphere boundary component of Ω_q. Two subsets X and Y of a metric space W have bounded Hausdorff distance if there exists an $\epsilon > 0$ so that $X \subset Nbhd_\epsilon(Y)$ and $Y \subset Nbhd_\epsilon(X)$. The infimum of such constants ϵ is called the *Hausdorff distance* between X and Y.

The second goal is to use Theorem 4.5, combined with Theorem 2.1, to prove Theorem B, i.e. that $PSL_2(\mathbf{Z}[\frac{1}{p}])$ and $PSL_2(\mathbf{Z}[\frac{1}{q}])$ are quasi-isometric if and only if $p = q$.

We use the notation $Nbhd_r(S)$ to be the r-neighborhood in Ω_p of a subset S of Ω_p. We say that a subset S of a metric space X has the *strong separation property* in X if there is a fixed $r > 0$ with the following property. For every $k > 0$, there are at least two connected components of $X - Nbhd_r(S)$ which contain metric balls of radius k. We will say that S *separates* X if S has the strong separation property in X. The constant r is called the *separation constant*.

A metric space X is called *uniformly contractible* if there is a function $\alpha : \mathbf{R}^+ \to \mathbf{R}^+$ with the following property. If $f : \Delta \to X$ is a map of a finite simplicial complex, and $f(\Delta) \subset B_r$, where $B_r \subset X$ is a metric r-ball, then $f(\Delta)$ is contractible in $B_{\alpha(r)}$, where α is independent of $dim(\Delta)$. Any contractible space admitting a cocompact group of isometries is uniformly contractible.

We will need the following coarse topology results which we state as special cases of Theorem 5.2 and Corollary 5.3 of [FS]. We are using the bounded geometry metric d on $\mathbf{R}^3$ described below which comes from choosing a proper embedding of the tree $T_p \to \mathbf{R}^2$. This allows us to consider $T_p \times \mathbf{R}$ as a subset of $(\mathbf{R}^3, d)$. In applying the results of [FS] we use the fact that $\mathbf{R}^3$ in this bounded geometry metric is uniformly contractible and contains spheres of arbitrarily large radius (the "expanding spheres" condition of [FS]).

Theorem 4.1 (Coarse Separation [FS]). *Suppose $\phi : (\mathbf{R}^3, d) \to Y$ is a $(K, C)-$quasi-isometric embedding of $(\mathbf{R}^3, d)$ into a uniformly contractible Riemannian manifold Y diffeomorphic to $\mathbf{R}^4$. Then $\phi(\mathbf{R}^3)$ separates Y, where the separation constant depends on (K, C).*

Corollary 4.2 (Packing Theorem [FS]). *Suppose $\phi : (\mathbf{R}^3, d) \to (\mathbf{R}^3, d)$ is a $(K, C)-$quasi-isometric embedding. Then ϕ is a $(K', C')-$quasi-isometry, for some constants (K', C') depending on (K, C).*

4.1 Separation

Let σ be any horosphere boundary component of Ω_p and $f : \Omega_p \to \Omega_q$ a quasi-isometry. In this section we show that the image $f(\sigma)$ separates $\mathbf{H}^2 \times T_q$. To prove this, we extend the quasi-isometric embedding $f|_\sigma : \sigma \to \mathbf{H}^2 \times T_q$ to a quasi-isometric embedding $\widehat{f} : (\mathbf{R}^3, d) \to \mathbf{H}^2 \times \mathbf{R}^2$

to which we can apply Theorem 4.1 (Coarse Separation). As above, d is the bounded geometry metric on $\mathbf{R}^3$ which comes from choosing a proper embedding of the tree $T_p \to \mathbf{R}^2$.

Consider σ as the complex X_p associated to $BS(1,p)$ and choose a homeomorphism $\beta : X_p \to T_p \times \mathbf{R}$. Let $\alpha_p : T_p \to \mathbf{R}^2$ be any proper embedding. Then we can consider X_p as a subset of $\mathbf{R}^3$ via the map $(\alpha \times Id) \circ \beta$. It is shown in [FM] that $\mathbf{R}^3$ can be given a bounded geometry metric d for which this map is an isometric embedding, as follows. The boundary of each connected component C of $\mathbf{R}^3 - X_p$ is topologically a plane. We use two coordinates (t,r) on this plane ∂C, where $t \in T_p$ and $r \in \mathbf{R}$. Choose a homeomorphism which identifies $\partial C \cup C$ with $\partial C \times [0,\infty)$. Then a point in C has three coordinates: (t,r,s) where t and r as above give a point in ∂C and $s \in [0,\infty)$. We use the product metric on each component.

The quasi-isometric embedding $f|_\sigma$ has image in $\mathbf{H}^2 \times T_q$. Analogous to the above situation, we choose a proper embedding $\alpha_q : T_q \to \mathbf{R}^2$ and view $\mathbf{H}^2 \times T_q$ as a subset of $\mathbf{H}^2 \times \mathbf{R}^2$ via the map $Id \times \alpha$. As above, we obtain a metric d' on $\mathbf{H}^2 \times \mathbf{R}^2$ for which this map is an isometric embedding. To apply Theorem 4.1 we use the fact that $(\mathbf{H}^2 \times \mathbf{R}^2, d')$ is diffeomorphic to $\mathbf{R}^4$, uniformly contractible and contains arbitrary large metric balls.

We can now extend the map $f|_\sigma$ to a quasi-isometric embedding

$$\widehat{f} : (\mathbf{R}^3, d) \to (\mathbf{H}^2 \times \mathbf{R}^2, d')$$

by

$$\widehat{f}(t,r,s) = (f(t,r), s).$$

Note that $f(t,r)$ is a point in $\mathbf{H}^2 \times T_q$, and so provides two coordinates. Also, $\widehat{f}|_\sigma = f$.

Proposition 4.3 (Separation). *Let $f : \Omega_p \to \Omega_q \subset \mathbf{H}^2 \times T_q$ be a $(K,C)-$quasi-isometry, and let $\sigma \subset \partial\Omega_p$ be any horosphere boundary component. Then $f(\sigma)$ separates $\mathbf{H}^2 \times T_q$, where the separation constant depends on (K,C).*

Proof. Extend $f|_\sigma$ as above to a quasi-isometric embedding

$$\hat{f} : (\mathbf{R}^3, d) \to (\mathbf{H}^2 \times \mathbf{R}^2, d').$$

It is understood that a path "avoiding" $f(\sigma)$ or $\widehat{f}(\mathbf{R}^3)$ stays outside the neighborhood $Nbhd_r(f(\sigma))$ or $Nbhd_{r'}(\widehat{f}(\mathbf{R}^3))$ for a constant r or r'. Also note that $f(\sigma) = \widehat{f}(\sigma)$.

The map $\widehat{f} : (\mathbf{R}^3, d) \to (\mathbf{H}^2 \times \mathbf{R}^2, d')$ satisfies the conditions of Theorem 4.1, so $\widehat{f}(\mathbf{R}^3)$ separates $\mathbf{H}^2 \times \mathbf{R}^2$.

Suppose that $\hat{f}(\mathbf{R}^3)$ separates $\mathbf{H}^2 \times \mathbf{R}^2$ and $\widehat{f}(\sigma)$ does not separate $\mathbf{H}^2 \times T_q$. This means that any two points in $(\mathbf{H}^2 \times T_q) - \widehat{f}(\sigma)$ can be

joined by a path avoiding $\widehat{f}(\sigma)$. We will show that if $\widehat{f}(\sigma)$ does not separate $\mathbf{H}^2 \times T_q$, any two points in $(\mathbf{H}^2 \times \mathbf{R}^2) - \widehat{f}(\mathbf{R}^3)$ can be connected by a path avoiding $\widehat{f}(\mathbf{R}^3)$, contradicting the fact that $\widehat{f}(\mathbf{R}^3)$ separates $\mathbf{H}^2 \times \mathbf{R}^2$. Let x_1 and x_2 be any two points in $(\mathbf{H}^2 \times \mathbf{R}^2) - \widehat{f}(\mathbf{R}^3)$. Each $x_i, i = 1, 2$, has coordinates (α_i, t_i, s_i) as above. Since $x_i \notin \widehat{f}(\mathbf{R}^3)$, each line $\{(\alpha_i, t_i, s) | s \in \mathbf{R}\}$ is not contained in $\widehat{f}(\mathbf{R}^3)$ by construction. When $s = 0$, each line gives a point in $\mathbf{H}^2 \times T_q$ not contained in $\widehat{f}(\sigma)$. Call these points β_i. Since $\widehat{f}(\sigma)$ does not separate $\mathbf{H}^2 \times T_q$, we can connect β_1 to β_2 by a path γ lying in $\mathbf{H}^2 \times T_q$ which avoids $\widehat{f}(\sigma)$ and hence $\widehat{f}(\mathbf{R}^3)$. So x_1 and x_2 are connected by the path

$$(\alpha_1, t_1, [0, s_2]) * \gamma * (\alpha_2, t_2, [0, s_1])^{-1}$$

which avoids $\widehat{f}(\mathbf{R}^3)$. Thus, if $\widehat{f}(\mathbf{R}^3)$ separates $\mathbf{R}^2 \times \mathbf{H}^2$ then $f(\sigma) = \widehat{f}(\sigma)$ separates $\mathbf{H}^2 \times T_q$. □

We now prove a lemma which shows that the space Ω_p with the neighborhood of any horosphere removed is path connected. This lemma is needed in the proof of Theorem 4.5.

Lemma 4.4. *Let r be any positive real number. For any horosphere boundary component σ of Ω_p, the space $\Omega_p - Nbhd_r(\sigma)$ is path connected.*

Proof. We will show that $\Omega_p - Nbhd_r(\sigma)$ is path connected for any $\sigma \subset \partial\Omega$ and for any r. For convenience we may assume that $\sigma = \sigma_\infty$.

For any horosphere $\tau \subset \partial\Omega_p$, consider the set of vertices of T_p where τ intersects $Nbhd_r(\sigma_\infty)$:

$$S_{r,\tau} = \{[L] \in Vert(T_p)|\ \tau|_{[L]} \cap Nbhd_r(\sigma_\infty) \neq \emptyset\}.$$

Since each τ has a closeness line with σ_∞, for large enough r the set $S_{r,\tau}$ is the neighborhood in T_p of a line $l_\tau \subset T_p$. There is one line l_τ for each horosphere $\tau \neq \sigma_\infty$. Since the horosphere boundary components of Ω_p are indexed by rationals and the set of lines of T_p is indexed by elements of $\mathbf{Q}_p$, there are lines $l \subset T_p$ such that no horosphere intersects $Nbhd_r(\sigma_\infty)$ over every point of l. Let l' be such a line.

Choose two points x and y in $\Omega_p - Nbhd_r(\sigma)$. If x and y lie in $\mathbf{H}^2 \times l'$, then there is a path connecting them. If this is not the case, let α be any path from x to a point $x' \in \mathbf{H}^2 \times l'$, and β any path from y to a point $y' \in \mathbf{H}^2 \times l'$. Let γ be a path in $\mathbf{H}^2 \times l'$ connecting x' and y'. The composite path $\beta^{-1} \circ \gamma \circ \alpha$ connects x to y and lies in $\Omega_p - Nbhd_r(\sigma)$, proving the theorem. □

4.2 Proof of the Boundary Detection Theorem

We now state and prove Theorem 4.5, which plays a role in the proofs of Theorems A and B.

Theorem 4.5 (Boundary Detection Theorem). *Let $f : \Omega_p \to \Omega_q$ be a $(K, C)-$quasi-isometry. There exist constants (K', C') depending on (K, C) and the spaces Ω_p and Ω_q, with the following property. For every horosphere boundary component σ of Ω_p, there is a horosphere boundary component τ of Ω_q so that $f|_\sigma : \sigma \to \tau$ is a $(K', C')-$quasi-isometry.*

The following lemmas form the two major components of the proof of Theorem 4.5. In these lemmas, $f : \Omega_p \to \Omega_q$ will be a $(K, C)-$quasi-isometry.

Lemma 4.6. *There exists a constant ϵ depending on (K, C) and the spaces Ω_p and Ω_q, with the following property. For every horosphere boundary component σ of Ω_p, there is a horosphere boundary component τ of Ω_q so that $\tau \subset Nbhd_\epsilon(f(\sigma))$.*

Proof. Consider a horosphere boundary component σ of Ω_p. We will show that any two points x_1 and x_2 of $\mathbf{H}^2 \times T_q$ which are outside $Nbhd_\delta(f(\sigma))$ (for some constant δ) can be connected by a path avoiding $Nbhd_\delta(f(\sigma))$. If the lemma is false, then any horosphere $\tau \subset \partial\Omega_q$ contains points not within ϵ of $f(\sigma)$. For the proper choice of ϵ, this will contradict the fact that $f(\sigma)$ separates $\mathbf{H}^2 \times T_q$.

Choose points $x_1, x_2 \notin Nbhd_\delta(f(\sigma))$. If $x_i \notin \Omega_q$, $i = 1, 2$, we can find a path connecting x_i to some $x'_i \in \Omega_q$ which avoids $Nbhd_\delta(f(\sigma))$. Hence we will assume that $x_i \in \Omega_q$.

Let f^{-1} be a coarse inverse to f and consider the points $y_i = f^{-1}(x_i) \subset \Omega_p$ for $i = 1, 2$. If δ is sufficiently large, these points lie outside $Nbhd_{\delta_1}(\sigma)$, for some δ_1. From Lemma 4.4 we know that there is a path γ between y_1 and y_2 which avoids $Nbhd_{\delta_1}(\sigma)$. Then for some ϵ, $f(\gamma)$ is a path in Ω_q connecting x_1 and x_2 but avoiding $Nbhd_\epsilon(f(\sigma))$. (Figure 7.) If our initial δ was large enough, then $\epsilon > \delta$ and we can make this argument with $x_1 \in \tau$, $x_1 \notin Nbhd_\delta(f(\sigma))$, contradicting the fact that $f(\sigma)$ separates $\mathbf{H}^2 \times T_q$. $\square$

Lemma 4.7. *There exists a constant ϵ' depending on (K, C) and the spaces Ω_p and Ω_q, with the following property. For every horosphere boundary component σ of Ω_p, there is a horosphere boundary component τ of Ω_q so that $f(\sigma) \subset Nbhd_{\epsilon'}(\tau)$.*

Proof. Consider a horosphere boundary component σ of Ω_p. From Lemma 4.6, there is a horosphere boundary component τ of Ω_q and a constant ϵ so that $\tau \subset Nbhd_\epsilon(f(\sigma))$. Define a map $\psi : \tau \to \sigma$ by

$\psi(y) = x \in \sigma$, where x is any point so that $f(x)$ is metrically closest to y. If there is more than one such point, choose randomly. From Lemma 4.6, we see that ψ differs from the coarse inverse f^{-1} of f by at most a constant. From Proposition 4.3 we know that $\psi(\tau)$ separates $\mathbf{H}^2 \times T_q$. So for some constant δ', the horosphere σ is contained in $Nbhd_{\delta'}(\psi(\tau))$, i.e. every point of σ is within a constant δ' of some point $x \in \psi(\tau)$ which maps to a point $f(x)$ within ϵ of a point of τ. By enlarging δ' to a constant ϵ', we see that $f(\sigma) \subset Nbhd_{\epsilon'}(\tau)$. Since all the horospheres are isometric, the constant ϵ' is independent of the choice of σ. $\square$

Proof of the Boundary Detection Theorem. Apply Lemma 4.7 to both f and f^{-1}. Compose f with a nearest point projection to obtain a quasi-isometric embedding $f : \sigma \to \tau$. As in §4.1, embed σ and τ isometrically in $(\mathbf{R}^3, d)$ and extend f to a quasi-isometric embedding $\hat{f} : (\mathbf{R}^3, d) \to (\mathbf{R}^3, d)$. (Recall that d was the bounded geometry metric on $\mathbf{R}^3$ described in §4.1.) From the Packing Theorem, this map is a $(K', C')-$quasi-isometry, where the pair (K', C') depends on (K, C). Then by the construction of $\hat{f}$, the map f must also be a $(K', C')-$quasi-isometry. $\square$

It is now clear that closeness lines are preserved under quasi-isometry.

4.3 Proof of Theorem B

Theorem 4.5 allows us to prove Theorem B.

Proof of Theorem B. If $PSL_2(\mathbf{Z}[\frac{1}{p}])$ and $PSL_2(\mathbf{Z}[\frac{1}{q}])$ are commensurable, then they are automatically quasi-isometric.

Let $f : PSL_2(\mathbf{Z}[\frac{1}{p}]) \to PSL_2(\mathbf{Z}[\frac{1}{q}])$ be a $(K, C)-$quasi isometry. Construct the spaces Ω_p and Ω_q corresponding to $PSL_2(\mathbf{Z}[\frac{1}{p}])$ and $PSL_2(\mathbf{Z}[\frac{1}{q}])$, respectively. Then f induces a quasi-isometry, also denoted f, from Ω_p to Ω_q. Theorem 4.5 allows us to restrict f to a quasi-isometry $\hat{f}$ on horospheres. In §2.3 we showed that a horosphere of $\mathbf{H}^2 \times T_p$ has the geometry of the group $BS(1, p)$. Hence $\hat{f}$ is a quasi-isometry from $BS(1, p)$ to $BS(1, q)$. According to Theorem 2.1, we must have $p = q$ for such a quasi-isometry to exist. $\square$

5 Theorems A and C

Every commensurator of $PSL_2(\mathbf{Z}[\frac{1}{p}])$ acts as a quasi-isometry of Ω_p. To prove Theorem A, we must show that by composing an element $f \in QI(PSL_2(\mathbf{Z}[\frac{1}{p}]))$ with a specific commensurator, we obtain a map which is a bounded distance from the identity map. It is Theorem 5.3 (stated below) that tells us which commensurator to choose for this

purpose. In the Appendix, we show that the commensurator group of $PSL_2(\mathbf{Z}[\frac{1}{p}])$ in $PSL_2(\mathbf{R}) \times PSL_2(\mathbf{Q}_p)$ is $PSL_2(\mathbf{Q})$, where PSL_2 is viewed as an algebraic group as in §2.2.

In all that follows, we assume that $f : \Omega_p \to \Omega_p$ is a $(K, C)-$quasi-isometry which has been changed by a bounded amount using the "connect the dots" procedure so that it is continuous. (See, e.g. [FS].) Since $Comm(PSL_2(\mathbf{Z}[\frac{1}{p}])) = PSL_2(\mathbf{Q})$ acts transitively on pairs of distinct points of $\mathbf{R} \cup \{\infty\}$, we can assume that f has been composed with a commensurator so that $f(\sigma_\infty) = \sigma_\infty$ and $f(\sigma_0) = \sigma_0$. (Note that these horospheres are not necessarily fixed pointwise.)

5.1 Action Rigidity

We will now define a boundary of the space $\mathbf{H}^2 \times T_p$. Let t be the common endpoint of any two lines in T_p and consider $\partial_\infty(\mathbf{H}^2 \times t) - \{\infty\} \cong \mathbf{R}$. Recall from §2.3 that we can consider $\sigma_\infty \subset \partial\Omega_p$ as a quasi-isometrically embedded copy of the 2-complex X_p associated to $BS(1, p)$. So we can refer to the lower boundary $\partial_l(\sigma_\infty)$ of σ_∞ (resp. the upper boundary $\partial^u(\sigma_\infty)$) as the lower (resp. upper) boundary of X_p. The inclusion $i : \sigma_\infty \to \mathbf{H}^2 \times T_p$ induces an identification between $\partial_l(\sigma_\infty)$ and the copy of $\mathbf{R}$ described above. Since $f(\sigma_\infty) = \sigma_\infty$, the quasi-isometry f induces bilipschitz maps f_l and f^u on $\partial_l(\sigma_\infty)$ and $\partial^u(\sigma_\infty)$ of σ_∞, respectively. The restriction of f to $\mathbf{R} = \partial_\infty(\mathbf{H}^2 \times t) - \{\infty\}$ determines the permutation of the horospheres of $\mathbf{H}^2 \times T_p$ under the map f. Hence, from the identification induced by the inclusion map i, the map f_l is exactly the map which determines the permutation of the horospheres under f.

Let Δ_Q denote the lattice in $\mathbf{R} \times \mathbf{Q}_p$ given by the diagonal $\{(a, a) | a \in \mathbf{Q}\} \subset \mathbf{R} \times \mathbf{Q}_p$, and Δ the sublattice $\{(b, b) | b \in \mathbf{Z}[\frac{1}{p}]\}$. Clearly $\Delta \subset \Delta_Q$. We view of the first coordinate of the pair $(b, b) \in \Delta_Q$ as $b \in \mathbf{Q} \subset \mathbf{R}$ denoting the basepoint of a horosphere of $\mathbf{H}^2 \times T_p$, and the second coordinate $b \in \mathbf{Q} \subset \mathbf{Q}_p$ as the point in $\mathbf{Q}_p$ determined by the closeness line of σ_b and σ_∞. Using the identification described above, we see that the map induced by f on $\mathbf{Q} \subset \mathbf{R}$ is exactly $f_l|_{\mathbf{Q}}$. We can view $\mathbf{Q} \subset \mathbf{Q}_p$ as a subset of $\partial^u(\sigma_\infty)$. Since $f(\sigma_\infty) = \sigma_\infty$, the map induced by f on $\mathbf{Q} \subset \mathbf{Q}_p$ is exactly $f^u|_{\mathbf{Q}}$. The maps $f^u|_{\mathbf{Q}}$ and $f_l|_{\mathbf{Q}}$ are identical because closeness lines are preserved under quasi-isometry and $f(\sigma_\infty) = \sigma_\infty$. Hence f induces a map of Δ_Q given by $(f_l|_{\mathbf{Q}}, f^u|_{\mathbf{Q}})$, and we will use a single coordinate for points of Δ_Q. Let ϕ denote the common restriction of f_l and f^u to $\mathbf{Q}$. Then ϕ is K_0-bilipschitz, for some constant K_0 depending on the pair (K, C).

Let H be the cyclic group generated by the matrix $\begin{pmatrix} p & 0 \\ 0 & \frac{1}{p} \end{pmatrix}$. We make the following definitions relating to a group-invariant diameter

function which will allow us to state Theorem 5.3 (Action Rigidity).

Definitions.

1. For any subset S of $\mathbf{R} \times \mathbf{Q}_p$ and H as above, define the H-invariant diameter of S by

$$\delta_H(S) = \inf_{T \in H} diam(T(S)).$$

For the remainder of this paper we will write δ for δ_H.

2. For subsets S_1, S_2 of $\mathbf{R} \times \mathbf{Q}_p$, we say that the map $\phi : S_1 \to S_2$ is *quasi-adapted* to δ if there exists a map $\alpha : \mathbf{N} \to \mathbf{N}$ such that for any compact set V,

$$\delta(V) \leq k \Rightarrow \delta(\phi(V)) \leq \alpha(k)$$

and

$$\delta(\phi(V)) \leq k \Rightarrow \delta(V) \leq \alpha(k).$$

3. Let S be a subset of Δ_Q. We say that S has *bounded height* if $S \subset \frac{1}{M}\Delta$ for some $M \in \mathbf{Z}^+$ with $(M,p) = 1$.

4. A bijection $\phi : \Delta_Q \to \Delta_Q$ is said to be *quasi-integral* if both ϕ and ϕ^{-1} take sets of bounded height to sets of bounded height.

5. A bijection $\phi : \Delta_Q \to \Delta_Q$ is said to be *quasicompatible* with H if ϕ is quasi-integral and, when restricted to sets of bounded height, both ϕ and ϕ^{-1} are quasi-adapted to δ.

In a sequence of lemmas we show that the bilipschitz map $\phi : \Delta_Q \to \Delta_Q$ obtained from the original quasi-isometry f is quasi-compatible with H.

Associate to each point of Δ_Q the horosphere boundary component of Ω_p based at that point. Then the action of Γ_∞ on $\partial\Omega_p$ induces an action on Δ_Q. In particular, this action preserves denominators, i.e. $\Gamma_\infty \cdot \frac{1}{M}\Delta \subset \frac{1}{M}\Delta$. Consider the diameter function on subsets S of Δ_Q defined by

$$\delta_\Gamma(S) = \inf_{T \in \Gamma_\infty} diam\ T(S).$$

Also consider the following diameter function, with S as above. Let $\delta_B(S)$ be the diameter in Ω_p of the smallest metric ball which intersects all horosphere boundary components based at points of S. We now show that these two diameter functions are *quasi-identical* when restricted to $\frac{1}{M}\Delta$, i.e. if $\delta_\Gamma(S)$ is small for a subset S of Δ_Q, then $\delta_B(S)$ is bounded and vice versa.

Lemma 5.1. *The restrictions of* δ_Γ *and* δ_B *to* $\frac{1}{M}\Delta$ *are quasi-identical.*

Proof. Let S be a subset of Δ_Q so that $\delta_B(S)$ is small, say $\delta_B(S) = \epsilon$. Then there exists a point $x \in \Omega_p$ which is within ϵ of all horosphere boundary components based at points of S. Since $\Omega_p/PSL_2(\mathbf{Z}[\frac{1}{p}])$ is compact by construction, there are only a finite number of choices for the set S (modulo $\mathrm{Aut}(\Delta)$). Thus $\delta_\Gamma(S)$ must be bounded.

Now suppose $\delta_\Gamma(S)$ is small. By compactness there are only finitely many choices for S (modulo $\mathrm{Aut}(\Delta)$). Hence $\delta_B(S)$ is bounded. $\square$

Lemma 5.2. *For every $M \in \mathbf{Z}^+$, there exists $M' \in Z^+$ depending on the bilipschitz constant K_0 of ϕ and the space Ω_p, so that $\phi(\frac{1}{M}\Delta) \subset \frac{1}{M'}\Delta$.*

Proof. Consider the action of $\Gamma_\infty = Stab_{PSL_2(\mathbf{Z}[\frac{1}{p}])}(\infty)$ on $\frac{1}{M}\Delta$. This action preserves denominators, i.e. $\Gamma_\infty \cdot \frac{1}{M}\Delta \subset \frac{1}{M}\Delta$. In particular, under this action $\frac{1}{M}\Delta/_{\Gamma_\infty}$ consists of a finite set of points, which we view as a finite collection of horospheres $\sigma_1, \cdots \sigma_n \subset \partial\Omega_p$. Choose a point $x \in \sigma_\infty$ and consider the smallest metric ball in Ω_p based at x which intersects all of the σ_i. Let ϵ be the radius of this ball. There is an ϵ' (depending on ϵ) so that the ϵ' ball around $f(x)$ must intersect all of the $f(\sigma_i)$. Thus there are only a finite number of choices of horospheres in $\partial\Omega_p$ for the images $f(\sigma_i)$. It follows that there is a number $M' \in \mathbf{N}$ so that the collection of horospheres $\{f(\sigma_i)\}$ is based at points in $\frac{1}{M'}\Delta$. Since Γ_∞ preserves denominators, the image of σ_α (for any $\alpha \in \frac{1}{c}\Delta$) must be a horosphere based at a point of $\frac{1}{M'}\Delta$. This is equivalent to saying that $\phi(\frac{1}{M}\Delta) \subset \frac{1}{M'}\Delta$. $\square$

Lemma 5.2 shows that ϕ is quasi-integral, and since ϕ is a bilipschitz map of both $\mathbf{R}$ and $\mathbf{Q}_p$, it is quasi-adapted to δ. Hence ϕ is quasicompatible with H.

We can now state Theorem 5.3. We say that a map $\psi : \mathbf{R} \times \mathbf{Q}_p \to \mathbf{R} \times \mathbf{Q}_p$ is affine if its restriction to each factor is affine. This theorem is an $2-$dimensional S-arithmetic version of the Action Rigidity Theorem of R. Schwartz [S1].

Theorem 5.3 (Action Rigidity). *Let $\Delta_Q \subset \mathbf{R} \times \mathbf{Q}_p$ be the lattice $\{(a, a) | a \in \mathbf{Q}\}$ and let H be the group generated by the matrix $\begin{pmatrix} p & 0 \\ 0 & \frac{1}{p} \end{pmatrix}$. Then any bilipschitz map $\phi : \Delta_Q \to \Delta_Q$ which is quasi-compatible with H is the restriction of an affine map of $\mathbf{R} \times \mathbf{Q}_p$.*

5.2 The Parallelogram Lemma

The key lemma in the proof of Theorem 5.3 is Lemma 5.4 (Parallelogram Lemma). Let Δ_Q be as in §5.1 and let $\phi : \Delta_Q \to \Delta_Q$ be quasi-compatible with H and K_0-bilipschitz. We now make some preliminary definitions.

Definition. Let $M \in \mathbf{Z}^+$. A *parallelogram* P in $\frac{1}{M}\Delta$ is a quadruple of points $\begin{bmatrix} a & b \\ c & d \end{bmatrix}$ with $a, b, c, d \in \frac{1}{M}\Delta$ satisfying $a - c = b - d$.

We say that $\phi(P)$ is the quadruple $\begin{bmatrix} \phi(a) & \phi(b) \\ \phi(c) & \phi(d) \end{bmatrix}$. The goal of Lemma 5.4 is to determine when $\phi(P)$ is itself a parallelogram. We now define two quantities associated to a parallelogram P which will be invariant under the group action and translation.

Definition. Let $P = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ be a parallelogram. The *H-invariant perimeter* of P is given by

$$per(P) = \delta(a \cup b) + \delta(a \cup c).$$

The *shape* of P is given by

$$shape(P) = |\nu(b - a) - \nu(c - a)|$$

where $\nu(p^n \frac{x}{y}) = n$ is the $p-$adic valuation on $\mathbf{Q}_p$. For any $T \in H$ and $x \in \Delta_Q$ we have $per(P) = per(T(P) + x)$ and $shape(P) = shape(T(P) + x)$.

Let $P = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ be a parallelogram such that $\phi(P) = \begin{bmatrix} a' & b' \\ c' & d' \end{bmatrix}$ is again a parallelogram. Since ϕ is quasi-compatible with H, there exists a constant D (depending on $per(P)$ and K) such that

$$\delta(a' \cup b') + \delta(a' \cup c') \leq D.$$

By symmetry we also have $\delta(c' \cup d') + \delta(b' \cup d') \leq D$. From lemma 5.2 we obtain a constant k such that $\phi(\frac{1}{M}\Delta) \subset \frac{1}{k}\Delta$.

We now describe the points $x \in \frac{1}{k}\Delta,\ (k, p) = 1$, satisfying $\delta(0 \cup x) \leq D$.

$$S_{k,D} = \{x \in \frac{1}{k}\Delta | x \neq 0 \ and \ \delta(0 \cup x) \leq D\}.$$

The set $S_{k,D}$ is the orbit under G of a finite set of points of $\frac{1}{k}\Delta$, denoted $\{\frac{a_i}{kp^{r_i}}\}_{i \in I}$, where $(a_i, p) = 1$. Since δ is invariant under translation, the points of $\frac{1}{k}\Delta$ within D of some point $y \in \frac{1}{k}\Delta$ are given by $S_{k,D} + y$.

We use the following notation in the statement of Lemma 5.4. Let

$$B_1 = max_{i,j \in I}\{a_i - a_j\},\ B_2 = max_{i \in I}\{a_i\},$$

$$B_3 = max_{i,j \in I}\{|\nu(a_i - a_j)|, |\nu(a_i + a_j)|\}$$

and

$$B_4 = max_{i,j,k \in I}\{\nu(a_i + a_j - a_k)\}.$$

Lemma 5.4 (Parallelogram Lemma). *Let $\phi : \Delta_Q \to \Delta_Q$ be quasi-compatible with H and K_0-bilipschitz. Let $\log_p(K_0) = R$ and $L \in$ $\mathbf{N}$, $(L,p) = 1$. If P is a parallelogram in $\frac{1}{L}\Delta$ with*

$$per(P) \leq L \text{ and } shape(P) > s_0$$

where

$$s_0 = max\{2\log_p(B_1 + B_2) + 2R,\ 3B_3^2 + 2R,\ 2B_4 + 2R\}$$

then $\phi(P)$ is also a parallelogram.

Proof. Let $P = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ and $\phi(P) = \begin{bmatrix} a' & b' \\ c' & d' \end{bmatrix}$. We are assuming that $\phi(0) = 0$, so without loss of generality translate P so that $a = 0$ and hence $a' = 0$. As stated above, there is a constant D so that $\delta(0 \cup b') + \delta(0 \cup c') \leq D$ and $\delta(b' \cup d') + \delta(c' \cup d') \leq D$.

We can write $b' = p^n \frac{x}{k}$ and $c' = p^m \frac{y}{k}$ where $x = a_i$ and $y = a_j$ for some a_i, a_j as above. Then d' can be expressed in one of two ways. It is $\delta-$close to both b' and c', hence of the form $p^n \frac{x}{k} + p^N \frac{z}{k}$ and also $p^m \frac{y}{k} + p^M \frac{w}{k}$ where $z = a_i,\ w = a_j$ for some $i, j \in I$. Setting these expressions equal and clearing denominators yields

$$p^n x + p^N z = p^m y + p^M w. \qquad (*)$$

Note that x, y, z, w are all relatively prime to p.

We first obtain a lower bound on $n - m$, with n, m as above. We can write $b = p^{n_0} \frac{x_0}{L}$ and $c = p^{m_0} \frac{y_0}{L}$, with $(x_0, p) = 1$ and $(y_0, p) = 1$, chosen so that $shape(P) = n_0 - m_0$. We know by assumption that $n_0 - m_0 > s_0$. Since ϕ is a K_0-bilipshitz map on $\mathbf{Q} \subset \mathbf{Q}_p$ and $\phi(0) = 0$, letting $R = \log_p(K_0)$ we obtain

$$p^{-n_0 - R} \leq |\phi(p^{n_0} \frac{x_0}{L})|_p = |p^n \frac{x}{k}|_p \leq p^{-n_0 + R}.$$

Hence $n_0 - R \leq n \leq n_0 + R$; similarly, $m_0 - R \leq m \leq m_0 + R$. Thus,

$$(n_0 - m_0) - 2R \leq n - m \leq (n_0 - m_0) + 2R.$$

Using the fact that $n_0 - m_0 > s_0$, we see that

$$n - m \geq s = max\{2\log_p(B_1 + B_2),\ 3B_3^2,\ 2B_4\}.$$

Consider again the expression $(*)$ for d'. First we assume that not all exponents are equal. If all the exponents were equal, then n would equal m, which is impossible since $n - m > s > 0$. Suppose first that $n > N$. Since the $p-$norms of both sides must be equal, we know that $N = min\{m, M\}$.

Case 1. Suppose $N = m$ and $M \neq m$. Then $(*)$ simplifies to

$$p^{n-m}x = p^{M-m}w + (y - z).$$

We know that $n - m > s$. Suppose $y - z \neq 0$. The highest power of p dividing the right hand side of the equation is bounded by B_3. But by the choice of s, we have $n - m > B_3$; hence this situation cannot occur.

If by chance $M - m = B_3$, and $y - z = p^{B_3}t$ for some integer t, then note that $\nu(w + t)$ is also bounded. If this is the case, change the initial value of s_0 by the quantity $\nu(w + t)$, and we may assume that we are not in this case.

So we must have $y - z = 0$, i.e. $z = y$ and $p^{n-m}x = p^{M-m}w$. Thus $n = M$ and $x = w$, and $\phi(P)$ is a parallelogram.

Case 2. Suppose $N = M$ and $M \neq m$. Then $(*)$ simplifies to

$$p^{m-M}(p^{n-m}x - y) = w - z.$$

Since $m - M > 0$ and $w - z \leq B_1$, our choice of s_0 insures that $p^{m-M}(p^{n-m}x - y) > B_1$, a contradiction.

Case 3. Suppose that $N = M = m$. Then $(*)$ simplifies to

$$p^{n-m}x = y + z - w.$$

By our choice of s_0, the exponent $m - n > B_4$, a contradiction.

Now suppose that $n < N$. Then we have $n = min\{m, M\}$. Since $n - m > s > 0$, we cannot have $n = m$.

Case 4. Suppose that $n = M$. But then $M < m < n = M$, a contradiction.

Lastly, we consider the case $n = N$.

Case 5. Suppose $n = N$. Then $(*)$ simplifies to

$$p^n(x + z) = p^m y + p^M w.$$

The quantity $x + z$ can take one of a finite number of values, so we can write $x + z = p^h f$ where $(f, p) = 1$ and h is bounded by B_3. Then we must have $n + h = min(m, M)$. If $n + h = m$, then $-h = n - m > s$ which cannot happen, since $|h| \leq B_3$ but $n - m > B_3$. If $n + h = M$, then $(*)$ becomes $p^{n-m+h}(f - w) = y$. Again by the choice of s_0, the left hand side of the equation is greater than the right hand side. $\square$

5.3 Proof of the Action Rigidity Theorem

We now prove Theorem 5.3.

Proof of Theorem 5.3. Fix $q \in \mathbf{N}$, $(q,p) = 1$ and let S be a generating set for $\frac{1}{q}\Delta$ containing $\frac{1}{q}$. Let s_0 be as in the Lemma 5.4, and let $H(S)$ denote the orbit of S under H. Given $x, y \in \frac{1}{q}\Delta$ we say that (x,y) is a distinguished pair if $x - y \in H(S)$. Let $C = max_{a \in H(S)}(q, 2\delta(0 \cup a))$.

We write $P((x,y),(z,w))$ if $P = \begin{bmatrix} x & y \\ z & w \end{bmatrix}$ is a parallelogram with $P \subset \frac{1}{q}\Delta$, $per(P) \leq C$ and $shape(P) \geq s_0$. Then by the Parallelogram lemma, $\phi(P)$ will also be a parallelogram, i.e.

$$\phi(x) - \phi(z) = \phi(y) - \phi(w).$$

We write $\overline{P}((x,y),(z,w))$ if there is a finite sequence of pairs (a_i, b_i) for $i = 0, \cdots, n$ with $(a_0, b_0) = (x,y)$ and $(a_n, b_n) = (z,w)$ such that $P((a_i,b_i),(a_{i+1},b_{i+1}))$. This means that we have a sequence of parallelograms between the two given pairs of points, each satisfying the conclusions of the parallelogram lemma, and with any two consecutive parallelograms in this sequence sharing a common side. Concatenating these intermediate parallelograms allows us to conclude that the parallelogram given by the original pairs also satisfies the conclusions of the Lemma 5.4.

We first need the following lemma.

Lemma 5.5. *Let $a \in \frac{1}{q}\Delta$ be arbitrary, and let (u,v) be a distinguished pair. Then $\overline{P}((u,v),(u+a,v+a))$.*

Proof. Consider $P_y = \begin{bmatrix} u & v \\ u+y & v+y \end{bmatrix}$ for any $y \in H(S)$. By construction, we have $per(P_y) \leq C$. Let $Y = Y(u,v) \subset H(S)$ denote those $y \in H(S)$ such that $shape(P_y) \geq s_0$. Let $\Sigma_0 Y$ be the sublattice in Δ_Q generated by Y and $\Sigma Y = \Sigma_0 Y \cap \frac{1}{q}\Delta$. We will first show that $\overline{P}((u,v),(u+x,v+x))$ for any $x \in \Sigma Y$ and then show that $\Sigma Y = \frac{1}{q}\Delta$. We prove this first assertion by induction. If $x \in Y$, it is certainly true that $P((u,v),(u+x,v+x))$. If $x \in \Sigma Y$, write $x = x' + y$ for $y \in Y$, and we have $\overline{P}((u,v),(u+x',v+x'))$ by the induction hypothesis. Consider the parallelogram

$$P = \begin{bmatrix} u+x' & v+x' \\ u+x'+y & v+x'+y \end{bmatrix} = \begin{bmatrix} u+x' & v+x' \\ u+x & v+x \end{bmatrix}.$$

We see that $shape(P) = shape(P_y) \geq s_0$. Also, $\delta(u+x' \cup u+x) = \delta(0 \cup y) \leq \frac{1}{2}C$ and $\delta(u+x' \cup v+x') = \delta(u \cup v) \leq \frac{1}{2}C$. So we have $per(P) \leq C$. Therefore we know that $P((u+x',v+x'),(u+x,v+x))$.

Now we show that $\Sigma Y = \frac{1}{q}\Delta$. We know that $shape(P_y) = |\nu(u-v) - \nu(y)|$. Let $\omega = \nu(u-v)$. Then $Y = \{y \in H(S) | |\omega - \nu(y)| \geq s_0\}$. Hence for large enough n, we have $\frac{1}{p^n}\frac{1}{q} \in Y$, so $\Sigma Y = \frac{1}{q}\Delta$, proving the lemma. □

Given $x \in \frac{1}{q}\Delta$, we can create a distinguished pair by taking $(x, x+a)$, for any $a \in S$. So for any $x, y \in \frac{1}{q}\Delta$ we know that $P((x, x+a), (y, y+a))$, or $\phi(x+a) - \phi(x) = \phi(y+a) - \phi(y)$. Since S generates $\frac{1}{q}\Delta$ we know that for any $x, y, z \in \frac{1}{q}\Delta$ we have $\overline{P}((x, x+z), (y, y+z))$ or

$$\phi(x+z) - \phi(x) = \phi(y+z) - \phi(y).$$

In particular, since $\phi(0) = 0$, we know that $\phi|_{\frac{1}{q}\Delta}$ is multiplication by a constant C_q. Let $q_1, q_2 \in \mathbf{N}$, with $(q_1, p) = (q_2, p) = 1$. Then we must have $C_{q_1 q_2} = C_{q_1} = C_{q_2}$. Hence there is a constant α such that the map $\phi|_{\Delta_Q}$ is multiplication by α. It follows that ϕ is the restriction of an affine map of $\mathbf{R} \times \mathbf{Q}_p$. $\square$

5.4 Proof of Theorem A

We now use Theorem 5.3 to prove Theorem A.

Proof of Theorem A. It is clear that every commensurator of $PSL_2(\mathbf{Z}[\frac{1}{p}])$ gives rise to a unique quasi-isometry of Ω_p. Given $f \in QI(PSL_2(\mathbf{Z}[\frac{1}{p}]))$, we will now choose a commensurator $g \in Comm(PSL_2(\mathbf{Z}[\frac{1}{p}]))$ so that the composition $g \circ f$ is a bounded distance from the identity map.

Let $f \in QI(PSL_2(\mathbf{Z}[\frac{1}{p}]))$. Then f induces a quasi-isometry from Ω_p to itself. Compose f with a commensurator so that $f(\sigma_\infty) = \sigma_\infty$ and $f(\sigma_0) = \sigma_0$. Combining Theorems 2.2 and 5.3, we know that f induces a map f_l on the lower boundary of σ_∞ which is multiplication by a constant α and determines the permutation of the horospheres under f.

Recall from §2.2 that we view $PSL_2(\mathbf{Q})$ as the $\mathbf{Q}-$points $G'_{\mathbf{Q}}$, where $G' = Ad(SL_2(\mathbf{C}))$. Compose f with the commensurator $g = Ad\begin{pmatrix} \frac{1}{\sqrt{\alpha}} & 0 \\ 0 & \sqrt{\alpha} \end{pmatrix}$. Then the permutation of the horospheres under $g \circ f$ is the identity permutation. We must now show that for any $m \in \Omega_p$, the image $f(m)$ lies in an ϵ-ball around m, where ϵ is independent of the choice of m.

The set of points within n units of any three distinct horospheres has bounded diameter, independent of the choice of horospheres. Also, any quasi-isometry f has the property that

$$hd(f(Nbhd(A) \cap Nhbd(B)), Nbhd(f(A) \cap Nbhd(B))) < \infty$$

where hd denotes Hausdorff distance. So if $x \in \Omega_p$ lies in the intersection of the n-neighborhoods of three horospheres, then there is a constant n' so that $f(x) \in B_{n'}(x)$. Thus $f \circ g$ is a bounded distance from the identity map, so the natural map $\Psi : Comm(PSL_2(\mathbf{Z}[\frac{1}{p}])) \to QI(PSL_2(\mathbf{Z}[\frac{1}{p}]))$ is an isomorphism. $\square$

5.5 Proof of Theorem C

We now prove Theorem C. The proof uses some standard techniques from the study of quasi-isometric rigidity for lattices in semisimple Lie groups. In addition to these techniques, Theorem A is applied, as well as the S-arithmetic superrigidity theorem of Margulis [M].

Proof of Theorem C. Since $PSL_2(\mathbf{Z}[\frac{1}{p}])$ is quasi-isometric to the space Ω_p, we get a quasi-isometry $f : \Gamma \to \Omega_p$. To obtain the exact sequence of the theorem, we will find a representation $\rho : \Gamma \to PSL_2(\mathbf{R}) \times PSL_2(\mathbf{Q}_p)$ with finite kernel so that $\rho(\Gamma)$ is a nonuniform lattice in $PSL_2(\mathbf{R}) \times PSL_2(\mathbf{Q}_p)$, hence commensurable to $PSL_2(\mathbf{Z}[\frac{1}{p}])$ by [M].

Since Γ acts on itself by isometries via left multiplication L_γ, for all $\gamma \in \Gamma$, we obtain a uniform family of quasi-isometries

$$f_\gamma = f \circ L_\gamma \circ f^{-1} : \Omega_p \to \Omega_p.$$

From Theorem A, we can think of f_γ as a commensurator, hence a bounded distance from an isometry. For each f_γ, compose with a bounded alteration B_γ to obtain a map

$$\rho : \Gamma \to Isom(\mathbf{H}^2 \times T_p).$$

First we show that ρ is a homomorphism. Suppose $\rho(\gamma)$ is a bounded distance from the identity isometry, i.e. $d_{\mathbf{H}^2 \times T_p}(x, \rho(\gamma) \cdot x) < \epsilon$ for all $x \in \mathbf{H}^2 \times T_p$. But then it follows that $\rho(\gamma)$ is the identity, hence ρ is a homomorphism. We must show that $\rho(\gamma)$ is a lattice and that ρ has finite kernel.

Since $f(\Gamma)$ is a net in Ω_p, $\rho(\Gamma) = \{f_\gamma\} = \{B_\gamma \circ f \circ L_\gamma \circ f^{-1} | \gamma \in \Gamma\}$ acts cocompactly on Ω_p. It follows that $\rho(\Gamma)$ acts on $\mathbf{H}^2 \times T_p$ with cofinite volume.

Choose a basepoint $x \in \Omega_p$ and consider $d_{\mathbf{H}^2 \times T_p}(x, \rho(\gamma) \cdot x)$. For finitely many $\gamma \in \Gamma$, $\rho(\gamma)$ moves x a bounded amount, i.e. there exists a constant $C' > 0$ so that $d_{\mathbf{H}^2 \times T_p}(x, \rho(\gamma) \cdot x) \leq C'$ for some finite set $\{\gamma_i\} \subset \Gamma$. (To find C' we are using the fact that $\{f_\gamma\}$ is a uniform family of quasi-isometries.) In particular, $d_{\mathbf{H}^2 \times T_p}(x, \rho(\gamma) \cdot x) = 0$ for only finitely many $\gamma \in \Gamma$. But if $d_{\mathbf{H}^2 \times T_p}(x, \rho(\gamma) \cdot x) > 0$, then $\rho(\gamma)$ is not the identity isometry. Hence ρ has finite kernel. The previous paragraph showed that $\rho(\Gamma)$ is discrete, so we can now construct the short exact sequence

$$1 \to N \to \Gamma \to \Lambda \to 1$$

where $N = ker(\rho)$ and $\Lambda = \rho(\Gamma)$ is a lattice in $PSL_2(\mathbf{R}) \times PSL_2(\mathbf{Q}_p)$.

To complete the proof, we apply the following theorem of Margulis [M] (Theorem 5.6), stated in the case $n = 2$, from which we conclude that the lattice Λ obtained above is commensurable to $PSL_2(\mathbf{Z}[\frac{1}{p}])$.

Theorem 5.6. *[M] Let p be a prime and $n \in N^+$, $n \geq 2$. If a subgroup Γ of $SL_2(\mathbf{Q})$ under the diagonal embedding in $SL_2(\mathbf{R}) \times SL_2(\mathbf{Q}_p)$ is a lattice in $SL_2(\mathbf{R}) \times SL_2(\mathbf{Q}_p)$, then the subgroups Γ and $SL_2(\mathbf{Z}[\frac{1}{p}])$ are commensurable.*

The proof of Theorem 5.6 uses Margulis' superrigidity theorem for S-arithmetic groups. Theorem 5.6 completes the proof of Theorem C. □

APPENDIX

We now compute the commensurator group $Comm(PSL_2(\mathbf{Z}[\frac{1}{p}]))$ of $PSL_2(\mathbf{Z}[\frac{1}{p}])$ in $PSL_2(\mathbf{R}) \times PSL_2(\mathbf{Q}_p)$.

Proposition. *The commensurator subgroup for $PSL_2(\mathbf{Z}[\frac{1}{p}])$ is given by*

$$Comm(PSL_2(\mathbf{Z}[\frac{1}{p}])) = PSL_2(\mathbf{Q})$$
$$= \{(a,a)|a \in PSL_2(\mathbf{Q})\} \subset PSL_2(\mathbf{R}) \times PSL_2(\mathbf{Q}_p).$$

Proof. Since any commensurator g must preserve Ω_p, hence preserve the horospheres, i.e. map $\mathbf{Q} \cup \{\infty\}$ to $\mathbf{Q} \cup \{\infty\}$, we must have $g \in PSL_2(\mathbf{Q})$. We now show that the commensurator subgroup, which is contained in $PSL_2(\mathbf{R}) \times PSL_2(\mathbf{Q}_p)$, is exactly the set $\{(g,g)|g \in PSL_2(\mathbf{Q})\}$.

First we show that for $b \in PSL_2(\mathbf{Q})$, the element $h = (Id, b)$ is not a commensurator of $PSL_2(\mathbf{Z}[\frac{1}{p}])$. We know that an element $(a,b) \in Comm(PSL_2(\mathbf{Z}[\frac{1}{p}]))$ must induce a quasi-isometry on the space Ω_p. Since $a \in PSL_2(\mathbf{Q}) \subset PSL_2(\mathbf{R})$ and $b \in PSL_2(\mathbf{Q}) \subset PSL_2(\mathbf{Q}_p)$, we can let a act by conjugation on the $\mathbf{H}^2$ factor and b by conjugation on T_p. So h acts by the identity on $\mathbf{H}^2$, hence $h(\sigma_\alpha) = \sigma_\alpha$, although σ_α is not fixed pointwise. In addition, $h(\sigma_\infty) = \sigma_\infty$, so the closeness line between σ_α and σ_∞ is also preserved. Since this is true for all $\alpha \in \mathbf{Q}$, if we view h as a map on $\mathbf{Q}_p$, then h fixes a copy of $\mathbf{Q} \subset \mathbf{Q}_p$. Hence when we extend h continuously to $\mathbf{Q}_p$, we see that it must be the identity.

We now show that if $a \in PSL_2(\mathbf{Q})$, then (a,a) is a commensurator of $PSL_2(\mathbf{Z}[\frac{1}{p}])$. A calculation shows that for any element

$$(g,g) \in PSL_2(\mathbf{Z}[\frac{1}{p}]) \times PSL_2(\mathbf{Z}[\frac{1}{p}]) \subset PSL_2(\mathbf{R}) \times PSL_2(\mathbf{Q}_p)$$

the expression

$$(a,a)(g,g)(a^{-1},a^{-1})$$

can be written as a matrix all of whose entries have bounded denominator, say bounded by d, meaning that each denominator is of the form dp^r for some $r \in \mathbf{Z}^+$. Thus the subgroup

$$(a,a)(PSL_2(\mathbf{Z}[\frac{1}{p}]), PSL_2(\mathbf{Z}[\frac{1}{p}]))(a^{-1}, a^{-1})$$

is of finite index in $PSL_2(\mathbf{Z}[\frac{1}{p}]) \times PSL_2(\mathbf{Z}[\frac{1}{p}])$. Hence (a,a) is a commensurator.

Now we will show that $Comm(PSL_2(\mathbf{Z}[\frac{1}{p}]))$ consists entirely of elements of the form (a,a), with $a \in PSL_2(\mathbf{Q})$. Suppose $(a,b) \in Comm(PSL_2(\mathbf{Z}[\frac{1}{p}]))$, with $a, b \in PSL_2(\mathbf{Q})$. We know that (a^{-1}, a^{-1}) is an element of $Comm(PSL_2(\mathbf{Z}[\frac{1}{p}]))$. Compose these two elements to form a new commensurator $(a,b) \circ (a^{-1}, a^{-1}) = (Id, ba^{-1})$. By the reasoning above, if this is a commensurator, then we must have $ba^{-1} = Id$, which implies that $b = a$, so our original commensurator must have been of the form (a,a) with $a \in PSL_2(\mathbf{Q})$. $\square$

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